University of Mannheim
School of Social Sciences
Mathematics for Political Scientists, Fall 2022
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Solutions Linear Algebra II

1. Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -6 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$

   Determine $x = \dfrac{\det((\mathbf{A}')^{-1}) \cdot \det(\mathbf{A}^3 \cdot \mathbf{B} \cdot \mathbf{C}) \cdot \det(((\mathbf{A}+\mathbf{B})^{-1})^4)}{\det(\mathbf{C}^{-1})}$

   We only need to calculate $\det(\mathbf{A}), \det(\mathbf{B}), \det(\mathbf{C}),$ and $\det(\mathbf{A}+\mathbf{B})$, which all happen to be equal to -12. The rest can be done using rules for determinants.

$$\begin{aligned} x &= \frac{\det((\mathbf{A}')^{-1}) \cdot \det(\mathbf{A}^3 \cdot \mathbf{B} \cdot \mathbf{C}) \cdot \det(((\mathbf{A}+\mathbf{B})^{-1})^4)}{\det(\mathbf{C}^{-1})} \\ &= \frac{(\det(\mathbf{A}))^{-1} \cdot (\det(\mathbf{A}))^3 \cdot \det(\mathbf{B}) \cdot \det(\mathbf{C}) \cdot (\det(\mathbf{A}+\mathbf{B}))^{-4}}{(\det(\mathbf{C}))^{-1}} \\ &= -12 \end{aligned}$$

2. Calculate the determinant of $\mathbf{A}_t = \begin{pmatrix} 1 & t & 0 \\ -2 & -2 & -1 \\ 0 & 1 & t \end{pmatrix}$, and show that it is never 0.

$$\begin{aligned} \det(\mathbf{A}_t) &= 1(-2)t + t(-1)0 + 0(-2)1 - 0(-2)0 - 1(-1)1 - t(-2)t \\ &= 2t^2 - 2t + 1 \\ t_{1|2} &= \frac{2 \pm \sqrt{4-8}}{4} \end{aligned}$$

   As the square root in the last equation is not defined, the determinant can never be zero.

3. Determine the inverse of the following matrices.

   $\mathbf{A} = (-\frac{1}{42}), \mathbf{B} = \begin{pmatrix} -1 & 8 \\ 2 & 4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 0 \\ 2 & 4 & 7 \end{pmatrix}$

$$\mathbf{A}^{-1}=(-42), \mathbf{B}^{-1}=-\frac{1}{20}\cdot\begin{pmatrix}4 & -8\\ -2 & -1\end{pmatrix}, \mathbf{C}^{-1}=\begin{pmatrix}7 & -8 & -3\\ 0 & \frac{1}{2} & 0\\ -2 & 2 & 1\end{pmatrix}$$

We obtain the inverse of $\mathbf{C}$ using the Gauss-Jordan algortithm.

$$\left(\begin{array}{ccc|ccc}1 & 4 & 3 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 2 & 4 & 7 & 0 & 0 & 1\end{array}\right) \quad \begin{array}{l}\mathrm{I}-2\cdot\mathrm{II}\\ \mathrm{II}:2\\ \mathrm{III}+2\cdot\mathrm{II}\end{array}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 3 & 1 & -2 & 0\\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0\\ 2 & 0 & 7 & 0 & -2 & 1\end{array}\right) \quad \mathrm{III}-2\cdot\mathrm{I}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 3 & 1 & -2 & 0\\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & -2 & 2 & 1\end{array}\right) \quad \mathrm{I}-3\cdot\mathrm{III}$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 7 & -8 & -3\\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & -2 & 2 & 1\end{array}\right)$$

4. $\mathbf{D}=\begin{pmatrix}2 & 4\\ 0 & 1\end{pmatrix}, \mathbf{F}=\begin{pmatrix}3 & -4\\ 1 & 0\end{pmatrix}, \mathbf{G}=\begin{pmatrix}1 & 4\\ -2 & -1\end{pmatrix}$

   Solve the matrix equation $-\mathbf{D}^3\cdot\mathbf{X}+\mathbf{D}\cdot\mathbf{F}=\mathbf{D}^2\cdot\mathbf{I}\cdot\mathbf{D}\cdot\mathbf{X}\cdot\mathbf{I}-\mathbf{D}\cdot\mathbf{F}\cdot\mathbf{G}$.

$$\begin{aligned}-\mathbf{D}^3\cdot\mathbf{X}+\mathbf{D}\cdot\mathbf{F} &= \mathbf{D}^2\cdot\mathbf{I}\cdot\mathbf{D}\cdot\mathbf{X}\cdot\mathbf{I}-\mathbf{D}\cdot\mathbf{F}\cdot\mathbf{G}\\ -\mathbf{D}^3\cdot\mathbf{X}+\mathbf{D}\cdot\mathbf{F} &= \mathbf{D}^3\cdot\mathbf{X}-\mathbf{D}\cdot\mathbf{F}\cdot\mathbf{G}\\ 2\mathbf{D}^3\mathbf{X} &= \mathbf{DF}(\mathbf{I}+\mathbf{G})\\ \mathbf{X} &= \frac{1}{2}(\mathbf{D}^2)^{-1}\cdot\mathbf{F}(\mathbf{I}+\mathbf{G})\end{aligned}$$

   Now, we only need to calculate the result, which is $\mathbf{X}=\begin{pmatrix}-1.25 & -4.5\\ 1 & 2\end{pmatrix}$.

5. $\mathbf{A}=\begin{pmatrix}4 & -7\\ 1 & 2\end{pmatrix}, \mathbf{B}=\begin{pmatrix}3 & -7\\ 1 & 1\end{pmatrix}, \mathbf{C}=\begin{pmatrix}4 & 2\\ 2 & -1\end{pmatrix}$

   Solve the matrix equation $3\cdot\mathbf{X}-(\mathbf{A}-\mathbf{B})^3\cdot\mathbf{X}-\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{I}=\mathbf{I}-\mathbf{C}'\cdot\mathbf{X}$.

$$\begin{aligned}3\cdot\mathbf{X}-(\mathbf{A}-\mathbf{B})^3\cdot\mathbf{X}-\mathbf{C}\cdot\mathbf{X}\cdot\mathbf{I} &= \mathbf{I}-\mathbf{C}'\cdot\mathbf{X}\\ (3\mathbf{I}-\underbrace{(\mathbf{A}-\mathbf{B})^3}_{\mathbf{I}}\underbrace{-\mathbf{C}+\mathbf{C}'}_{=0})\cdot\mathbf{X} &= \mathbf{I}\\ 2\mathbf{IX} &= \mathbf{I}\\ \mathbf{X} &= \frac{1}{2}\mathbf{I}\end{aligned}$$

6. Consider the following system of equations.

$$\begin{aligned}x_1-x_2-x_3 &= 1\\ -3x_1+4x_2+2x_3 &= -8\\ -2x_1+3x_2+x_3 &= -7\end{aligned}$$

(a) Rewrite the system of equations in matrix form.

$$\begin{pmatrix} 1 & -1 & -1 \\ -3 & 4 & 2 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ -7 \end{pmatrix}$$

(b) How many solutions does the system have? Why?

Setting up the extended matrix yields:

$$\begin{pmatrix} 1 & -1 & -1 & | & 1 \\ -3 & 4 & 2 & | & -8 \\ -2 & 3 & 1 & | & -7 \end{pmatrix}$$

After a few steps of the Gauss Jordan algorithm we reach:

$$\begin{pmatrix} 1 & -1 & -1 & | & 1 \\ -2 & 3 & 2 & | & -7 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

That shows that the matrix does not have full rank and includes the equation $0x_1 + 0x_2 + 0x_3 = 0$. Therefore, the system of equations has infinity many solutions.